hazard further conjecture as to the reason of this difference of behaviour of the instrument in the two elements we might suppose more definitely that the level error is caused by a warming of the eastern pier more rapidly than the western, whose conductivity is not so great; so that it expands vertically and affects the level, while at the same time the lateral expansion, being symmetrical about the central vertical plane which passes nearly through the pivots, would not affect the azimuth. The piers, or that part of them at least which supports the pivots, not being very thick, the lagging due to gradual conduction will not be very large, and we have the effect of difference of conductivity nearly free from any counteraction.

On the other hand, we must refer the azimuthal variation to changes of temperature probably at some depth below the surface of the soil, which do not happen to affect the level appreciably.

I have examined the variations of level and azimuth at other observatories for a few years to see if there is any similarity between these fluctuations for different instruments. It is somewhat difficult, without expending more time than I can well spare at present, to disentangle their annual variations from changes, secular and irregular, and generally larger than those noticed at Greenwich. I look forward with interest to a discussion by Mr. Finlay of the errors of the Cape Transit Circle promised in the last volume of "Cape Observations" (1879–81). Roughly speaking, there does not seem to be much similarity between the fluctuations of position of different instruments; for instance, the difference of epoch, which has been chiefly considered above, is very variable. And indeed Mr. Ellis pointed out in his paper above referred to how largely the variations of the present Transit Circle differ from those of Troughton's Transit, which occupied nearly the same site. As I have already said, however, in suggesting the above explanations of the changes in position of the Greenwich Transit Circle, we have at our disposal two causes which, combined in varying relative proportion, are capable of explaining fluctuations following or anticipating the temperature by very different periods of time.

On the Formulæ for Computing the Apparent Positions of a Satellite, and for Correcting the Assumed Elements of its Orbit. By A. Marth.

The methods of computation connected with the investigation of a satellite's orbit are comparatively simple and convenient, if they are duly selected to suit directly the coordinates furnished by the observations. If polar-coordinates have been observed, there exists apparently no good reason for not employing polar-coordinates also in the corresponding computations. The present

paper contains the formulæ, which seem to me the most advantageous and handy for computing the apparent positions and the corrections of the assumed elements in the chief cases which come under consideration.

Referring the plane of the satellite's orbit to the plane parallel to the terrestrial equator, let N denote the longitude or Right Ascension of the ascending node; J the inclination; Q the angle between the line of apsides and line of nodes or the departure of the lower apside from the node; $e = \sin\phi$ the eccentricity: μ, ϵ, v the mean, eccentric, and true anomaly of the satellite; $u = Q + v$ its true longitude in the orbit reckoned from the node; $u_0 = Q + \mu$ its mean longitude; r the radius vector; a the major semi-axis of the orbit or the satellite's linear mean distance from the planet's centre; $\frac{r}{a} = \rho$, so that ρ is the radius vector expressed in parts of the major semi-axis of the orbit. Further, let A, D denote the apparent Right Ascension and Declination of the planet's centre; α, δ those of the satellite; Δ the linear distance of the observer from the planet. If the origin of the coordinates is placed in the satellite instead of in the centre of the planet, the strict formulæ become somewhat simplified, but for the purpose α and δ must be known. In case approximately correct values of $\alpha - A$ and $\delta - D$ are available, like those of the three outer satellites of *Saturn*, published for some years past in the *Monthly Notices*, or if they can be derived directly from the observations, it will be worth while to take advantage of the favourable circumstance.

As most of the observed positions of satellites consist of measures of position-angles p and distances s, or are given in polar-coordinates, the formulæ for their proper computation may first be taken into consideration.

If the Earth's position, as seen from the satellite, is referred to the plane of the satellite's orbit by the longitude U + 180° and latitude B, and if P is the position-angle of the pole of the orbit (in Right Ascension N − 90° and Declination 90° − J) at the geocentric place of the satellite, the values of U, B, P are found from those of N, J, α, δ by means of the equations:

$$\cos B \sin U = \cos\delta \sin(\alpha - N)\cos J + \sin\delta \sin J$$
$$\cos B \cos U = \cos\delta \cos(\alpha - N)$$
$$\sin B = \cos\delta \sin(\alpha - N)\sin J - \sin\delta \cos J$$
$$\cos B \sin P = \quad -\cos(\alpha - N)\sin J$$
$$\cos B \cos P = \sin\delta \sin(\alpha - N)\sin J + \cos\delta \cos J$$

or by equivalent formulæ.

The satellite's orbital longitude $u = Q + v$, and also ρ being deduced from the mean anomaly μ, either by the usual formulæ:

$$\epsilon - e\omega^{\circ} \sin\epsilon = \mu \qquad \omega^{\circ} = 57^{\circ}\!.296 = \frac{1}{\text{arc } 1^{\circ}}$$
$$\rho \sin v = \sin\epsilon \cos\phi$$
$$\rho \cos v = \cos\epsilon - e$$

or by the series for the equation of the centre $v-\mu$ and $\log \rho$, the values of the position-angle p and of the distance s corresponding to u are found by means of the equations

$$\sin \sigma \sin (p-\mathrm{P}) = \sin (u-\mathrm{U})$$
$$\sin \sigma \cos (p-\mathrm{P}) = \cos (u-\mathrm{U}) \sin \mathrm{B}$$
$$\sin s = \frac{a}{\Delta} \cdot \rho \sin \sigma$$

in which σ is the arc on the sphere between the geocentric and planetocentric place of the satellite, or, without the introduction of σ, by

$$\sin s \sin (p-\mathrm{P}) = \frac{a}{\Delta} \rho \sin (u-\mathrm{U})$$

$$\sin s \cos (p-\mathrm{P}) = \frac{a}{\Delta} \rho \cos (u-\mathrm{U}) \sin \mathrm{B}$$

The position-angle p thus computed refers to the Declination-circle passing through the satellite. In case the observed position-angle refers to the Declination-circle passing through the point midway between the planet's centre and the satellite, a correction $= -\frac{1}{2} s \sin p \tan \delta$ must, when a strict comparison is required, be added to the computed p.

If A, D are substituted for a, δ in the formulæ for U, B, P, so that their ensuing values refer to the centre of the planet instead of the satellite, the correction of the computed p will be $= + \frac{1}{2} s \sin p \tan \mathrm{D}$, the computed σ (which must be taken between 90° and 180°, when $\cos (u-\mathrm{U})$ is negative) represents the arc between the geocentric place of the planet and the planetocentric place of the satellite, and the value of s must be found indirectly from the equation

$$\sin s = \frac{a}{\Delta} \rho \sin (\sigma - s),$$

if the employment of the equations

$$\Delta_1 \sin s \sin (p-\mathrm{P}) = a\rho \sin (u-\mathrm{U})$$
$$\Delta_1 \sin s \cos (p-\mathrm{P}) = a\rho \cos (u-\mathrm{U}) \sin \mathrm{B}$$
$$\Delta_1 \cos s = a\rho \cos (u-\mathrm{U}) \cos \mathrm{B} + \Delta,$$

in which Δ_1 denotes the distance of the satellite from the earth, is to be avoided.

The equations of condition for correcting the assumed elements of the orbit gain in clearness and simplicity by the introduction of

$$\sin \sigma \sin \tau = \sin \mathrm{B}$$
$$\sin \sigma \cos \tau = \cos \mathrm{B} \sin (u-\mathrm{U})$$
$$\cos \sigma = \cos \mathrm{B} \cos (u-\mathrm{U})$$

which serve, at the same time, as a partial control upon the equations for finding p:

$$\sin\sigma\sin(p-\mathrm{P})=\sin(u-\mathrm{U})$$
$$\sin\sigma\cos(p-\mathrm{P})=\cos(u-\mathrm{U})\sin\mathrm{B}$$
$$\sin s=\frac{a}{\Delta}\,.\,\rho\sin\sigma$$

The differentiation of these equations gives:

$$\sin\sigma dp=\sin\tau\,.\,d(u-\mathrm{U})-\cos\tau\cos(u-\mathrm{U})\,d\mathrm{B}+\sin\sigma d\mathrm{P}$$
$$d\sigma=\cos\tau\,.\,d(u-\mathrm{U})+\sin\tau\cos(u-\mathrm{U})\,d\mathrm{B}$$

But

$$\cos\mathrm{B}d\mathrm{U}=-\cos\mathrm{U}\sin\mathrm{B}d\mathrm{J}-\cos\mathrm{P}\cos\delta d\mathrm{N}$$
$$\cos\mathrm{B}d\mathrm{P}=-\cos\mathrm{U}d\mathrm{J}\qquad+\sin\mathrm{U}\sin\mathrm{J}d\mathrm{N}$$
$$d\mathrm{B}=+\sin\mathrm{U}d\mathrm{J}\qquad-\cos\mathrm{U}\sin\mathrm{J}d\mathrm{N}.$$

Hence, after some reductions:

$$\sin\sigma dp=\sin\tau du-\cos\tau\sin u d\mathrm{J}+(\cos\sigma\cos p\cos\delta-\sin\sigma\sin\delta)d\mathrm{N}$$
$$d\sigma=\cos\tau du+\sin\tau\sin u d\mathrm{J}+\sin p\cos\delta d\mathrm{N}$$
$$\rho\sin\sigma\,.\,\omega^\circ\frac{ds}{s}=\rho\cos\sigma d\sigma+\sin\sigma\omega^\circ d\rho+\rho\sin\sigma\omega^\circ\,.\,\frac{da}{a}.$$

The differential expressions of u, the satellite's true longitude, with respect to its mean longitude $u_\circ$ and Q and ϕ, are:

$$\rho du=\frac{\cos\phi}{\rho}du_\circ+\left(\rho-\frac{\cos\phi}{\rho}\right)d\mathrm{Q}+(1+\rho\sec^2\phi)\sin v\,.\cos\phi d\phi$$
$$=(1+e\cos v)\sec\phi du_\circ-\{2\cos v\}\tan\phi d\mathrm{Q}+\{2\sin v\}\cos\phi d\phi$$

in which the coefficients in $\{\}$ brackets denote the following equivalents, among which the most convenient, according to facilities at hand, may be selected:

$$\{2\cos v\}=\cos v+\cos\epsilon\cos\phi+\tan\tfrac{1}{2}\phi$$
$$=\frac{(2+e\cos v)\cos v+e+\tan\frac{1}{2}\phi\cos^2\phi}{1+e\cos v}$$
$$=\left(2-\frac{e\cos v}{1+e\cos v}\right)\cos v+\frac{e\cos\phi+\tan\frac{1}{2}\phi}{1+e\cos v}$$
$$=(1+\rho\sec^2\phi)\cos v+\rho(\tan\phi+\tan\tfrac{1}{2}\phi\sec^2\phi)$$
$$\{2\sin v\}=\sin v+\sin\epsilon\sec\phi$$
$$=\frac{2+e\cos v}{1+e\cos v}\sin v$$
$$=\left(2-\frac{e\cos v}{1+e\cos v}\right)\sin v$$
$$=(1+\rho\sec^2\phi)\sin v.$$

The substitution of these expressions and of

$$\omega^\circ d\rho=\tan\phi\sin v du_\circ-\sin v\,.\tan\phi d\mathrm{Q}-\cos v\,.\cos\phi d\phi$$

in the equations for $\sin\sigma dp$ and ds leads to the equations be-

tween the variations of the computed coordinates and the variations of the assumed elements

$$\begin{aligned}
\rho \sin \sigma \delta p = {} & \sin \tau \, . \, \frac{\cos \phi}{\rho} \, . \, \delta u_0 \\
& - \sin \tau \, . \, \{2 \cos v\} \, \omega^{\circ} (\tan \phi + \delta e) \sin \delta Q \\
& + \sin \tau \, . \, \{2 \sin v\} \, \omega^{\circ} [(\tan \phi + \delta e) \cos \delta Q - \tan \phi] \\
& - \rho \cos \tau \sin u \delta J \\
& + \rho (\cos \sigma \cos p \cos \delta - \sin \sigma \sin \delta) \, \delta N
\end{aligned}$$

$$\begin{aligned}
\rho \sin \sigma \, . \, \omega^{\circ} \frac{\delta s}{s} = {} & (\cos \sigma \cos \tau \, . \, \frac{\cos \phi}{\rho} + \tan \phi \sin \sigma \sin v) \delta u_0 \\
& - (\cos \sigma \cos \tau \{2 \cos v\} - \sin \sigma \sin v) \, \omega^{\circ} (\tan \phi + \delta e) \sin \delta Q \\
& + (\cos \sigma \cos \tau \{2 \sin v\} - \sin \sigma \cos v) \, \omega^{\circ} [(\tan \phi + \delta e) \cos \delta Q - \tan \phi] \\
& + \rho \cos \sigma \sin \tau \sin u \, . \, \delta J \\
& + \rho \cos \sigma \sin p \cos \delta \, . \, \delta N \\
& + \rho \sin \sigma \, . \, \omega^{\circ} \frac{\delta a}{a} .
\end{aligned}$$

In order to guard against some possible oversight or error in the interpretation of the variations connected with the eccentricity if they are written in the form $\tan \phi \, \delta Q$ and $\cos \phi \, \delta \phi$, I have substituted in these equations for

$$\tan \phi \delta Q \; \ldots\ldots \; \omega^{\circ} (\tan \phi + \delta e) \sin \delta Q$$

and for

$$\cos \phi \delta \phi \; \ldots . \; \omega^{\circ} [(\tan \phi + \delta e) \cos \delta Q - \tan \phi]$$

expressions which represent the true meaning of these variations, and will lead to right deductions of the corrections of Q and e in all cases. If not thus interpreted the expressions $\tan \phi \, \delta Q$ and $\cos \phi \, \delta \phi$ or δe may obviously be misleading, and may give very incorrect results whenever the assumed values of Q and e require more than small corrections.

If, instead of δQ and δe, the variations of suitable functions of Q and e are to be introduced, the equations for $\rho \, du$ and $d\rho$ suggest the advantage of selecting for the purpose not $e \sin Q$ and $e \cos Q$, but $E \sin Q$ and $E \cos Q$, where E is such a function of e or ϕ that

$$\tan \phi dE = E de = E \cos \phi d\phi .$$

Putting accordingly

$$E = e \, . \sec^2 \tfrac{1}{2}\phi \, . \, c^{-2 \sin^2 \frac{1}{2}\phi} = 2 \tan \tfrac{1}{2}\phi \, . \, c^{-2 \sin^2 \frac{1}{2}\phi} \qquad (c \text{ basis of nat. log.})$$

the differential coefficients of u and ρ with respect to $E \sin Q$ and $E \cos Q$ will be

$$\begin{aligned}
\rho du = \, . \, . \, & - \frac{(2 + e \cos v) \cos u + (e \cos \phi + \tan \frac{1}{2}\phi) \cos Q}{1 + e \cos v} \, . \, \omega^{\circ} \, . \, \frac{\tan \phi}{E} \, . \, d(E \sin Q) \\
& + \frac{(2 + e \cos v) \sin u + (e \cos \phi + \tan \frac{1}{2}\phi) \sin Q}{1 + e \cos v} \, . \, \omega^{\circ} \, . \, \frac{\tan \phi}{E} \, . \, d(E \cos Q),
\end{aligned}$$

$$d\rho = \, . \, . \, . \, . - \sin u \, . \, \frac{\tan \phi}{E} \, . \, d(E \sin Q) - \cos u \, . \, \frac{\tan \phi}{E} \, . \, d(E \cos Q).$$

Hence the eccentricity terms in the equations of condition for δp and δs become

$$\rho \sin \sigma \delta p = \ldots - \sin \tau \cdot \frac{(2+e\cos v)\cos u + (e\cos\phi + \tan\frac{1}{2}\phi)\cos Q}{1+e\cos v} \cdot \omega^\circ \cdot$$

$$\frac{\tan\phi}{E} \cdot \delta(E\sin Q)$$

$$+ \sin\tau \cdot \frac{(2+e\cos v)\sin u + (e\cos\phi+\tan\frac{1}{2}\phi)\sin Q}{1+e\cos v} \cdot \omega^\circ \cdot$$

$$\frac{\tan\phi}{E} \cdot \delta(E\cos Q)$$

$$\rho \sin\sigma \frac{\delta s}{s} = \ldots - (\cos\sigma\cos\tau \cdot \frac{(2+e\cos v)\cos u + (e\cos\phi+\tan\frac{1}{2}\phi)\cos Q}{1+e\cos v}$$

$$+ \sin\sigma\sin u) \cdot \frac{\tan\phi}{E} \cdot \delta(E\sin Q)$$

$$+ (\cos\sigma\cos\tau \cdot \frac{(2+e\cos v)\sin u + (e\cos\phi+\tan\frac{1}{2}\phi)\sin Q}{1+e\cos v}$$

$$- \sin\sigma\cos u) \cdot \frac{\tan\phi}{E} \cdot \delta(E\cos Q).$$

Though in this way the corrections of the assumed values of Q and e might be properly found and mistaken deductions avoided, the way would be needlessly circuitous; for, by substituting in the foregoing expressions for $\delta(E\sin Q)$ and $\delta(E\cos Q)$ their values

$$(E+\delta E)\sin(Q+\delta Q) - E\sin Q, \text{ and}$$
$$(E+\delta E)\cos(Q+\delta Q) - E\cos Q,$$

and by eliminating Q, the expressions are obtained which are already given in the equations of condition, and which, while of equal strictness, are obviously preferable on account of their greater simplicity and directness. Moreover, they allow the motion of the apses, if approximately known, to be easily taken into account.

The equations of condition have yet to be so adjusted that the differences between the observed and the computed co-ordinates are measured by the same unit. It will depend on the character of the measured position-angles and distances whether it is preferable to express the differences in seconds of arc or in units corresponding to an arc of one degree on a circle of suitable radius, and also how far the ensuing rules are better modified or altered.

If $\sin a_0 = \frac{a}{\Delta_0}$, where Δ_0 is some conveniently fixed value of Δ (not differing greatly from the average of the distances Δ at which observations are taken), and if $\frac{\Delta_0}{\Delta} = \nu$, the angular distances s of the satellite will be

$$s = a_0 \nu \cdot \rho \sin\sigma.$$

Making

$$\frac{a_0}{57{\cdot}3} = \kappa$$

the adopted unit, the equations of conditions will be

$$\begin{aligned}
\frac{s}{a_0}\,\delta p = &+\nu\sin\tau\,.\,\frac{\cos\phi}{\rho}\,.\,\delta u_0 \\
&-\nu\sin\tau\,.\,\{2\cos v\}\,.\,\omega^\circ\,(\tan\phi+\delta e)\sin\delta Q \\
&+\nu\sin\tau\,.\,\{2\sin v\}\,.\,\omega^\circ\,[(\tan\phi+\delta e)\cos\delta Q-\tan\phi \\
&-\nu\rho\cos\tau\,.\,\sin u\,.\,\delta J \\
&+\nu\rho\,(\cos\sigma\cos p\cos\delta-\sin\sigma\sin\delta)\,.\,\delta N \\
\frac{\delta s}{\kappa} = &+\nu(\cos\sigma\cos\tau\,.\,\frac{\cos\phi}{\rho}+\tan\phi\sin\sigma\sin v)\,.\,\delta u_0 \\
&-\nu(\cos\sigma\cos\tau\,.\,\{2\cos v\}+\sin\sigma\sin v)\,.\,\omega^\circ\,(\tan\phi+\delta e)\sin\delta Q \\
&+\nu(\cos\sigma\cos\tau\,.\,\{2\sin v\}-\sin\sigma\cos v)\,.\,\omega^\circ\,[(\tan\phi+\delta e)\cos\delta Q-\tan\phi] \\
&+\nu\rho\cos\sigma\sin\tau\,.\,\sin u\,.\,\delta J \\
&+\nu\rho\cos\sigma\sin p\cos\delta\,.\,\delta N \\
&+\frac{s}{a_0}\,.\,\frac{\delta a_0}{\kappa}.
\end{aligned}$$

If the differences between the observed and computed coordinates are expressed in seconds of arc, the equations become

$$\begin{aligned}
\frac{s}{57{\cdot}3}\,\delta p = &+\nu\sin\tau\,.\,\frac{\cos\phi}{\rho}\,.\,\kappa\delta u_0 \\
&-\nu\sin\tau\,.\,\{2\cos v\}\,.\,a_0\,(\tan\phi+\delta e)\sin\delta Q \\
&+\nu\sin\tau\,.\,\{2\sin v\}\,.\,a_0\,[(\tan\phi+\delta e)\cos\delta Q-\tan\phi] \\
&-\nu\rho\cos\tau\,.\,\sin u\,.\,\kappa\delta J \\
&+\nu\rho\,(\cos\sigma\cos p\cos\delta-\sin\sigma\sin\delta)\,.\,\kappa\delta N \\
\delta s = &+\nu\,(\cos\sigma\cos\tau\,.\,\frac{\cos\phi}{\rho}+\tan\phi\sin\sigma\sin v)\,.\,\kappa\delta u_0 \\
&-\nu\,(\cos\sigma\cos\tau\,.\,\{2\cos v\}+\sin\sigma\sin v)\,.\,a_0\,(\tan\phi+\delta e)\sin\delta Q \\
&+\nu\,(\cos\sigma\cos\tau\,.\,\{2\sin v\}-\sin\sigma\cos v)\,.\,a_0\,[(\tan\phi+\delta e)\cos\delta Q-\tan\phi] \\
&+\nu\rho\cos\sigma\sin\tau\,.\,\sin u\,.\,\kappa\delta J \\
&+\nu\rho\cos\sigma\sin p\cos\delta\,.\,\kappa\delta N \\
&+\frac{s}{a_0}\,.\,\delta a_0.
\end{aligned}$$

The formulæ are here written out in full. Abbreviated notations for the quantities which are to be determined and for their coefficients will, of course, be substituted in actual computations.

The preceding equations are valid for orbits of any assumed

ellipticity. For assumed circular elements they get considerably simplified, and become

$$\frac{s}{a_0}\delta p = +\nu \sin\tau \,.\, \delta u$$
$$-\nu \sin\tau \,.\, \cos u \,.\, 2\phi \sin Q$$
$$+\nu \sin\tau \,.\, \sin u \,.\, 2\phi \cos Q$$
$$-\nu \cos\tau \,.\, \sin u \,.\, \delta J$$
$$+\nu (\cos\sigma \cos p \cos\delta - \sin\sigma \sin\delta) \,.\, \delta N$$

$$\frac{\delta s}{\kappa} = +\nu \cos\sigma \cos\tau \,.\, \delta u$$
$$-\nu (\cos\sigma \cos\tau \,.\, \cos u + \tfrac{1}{2}\sin\sigma \sin u) \,.\, 2\phi \sin Q$$
$$+\nu (\cos\sigma \cos\tau \,.\, \sin u - \tfrac{1}{2}\sin\sigma \cos u) \,.\, 2\phi \cos Q$$
$$+\nu \cos\sigma \sin\tau \,.\, \sin u \,.\, \delta J$$
$$+\nu \cos\sigma \sin p \cos\delta \,.\, \delta N$$
$$+\frac{s}{a_0} \,.\, \frac{\delta a_0}{\kappa}.$$

They must be multiplied by κ if the differences are to be expressed in seconds.

The orbital longitudes u, U, Q employed in the preceding formulæ are reckoned from the node N, and this reckoning is most suitable whenever the inclination J is considerable. But if J is only of moderate amount, as in the case of the orbits of *Saturn's* satellites, the determination of the nodal point is considerably uncertain, and this uncertainty rapidly increases with the decrease of J. By adding the longitude of N to the longitudes reckoned from N the uncertainty is evaded, and the longitudes in the orbit start from a properly fixed point.

Putting $u+N=l$ and $U+N=L$, the formulæ for finding p and the sin and cos of σ and τ require merely the substitution of $l-L$ for $u-U$:

$$\sin\sigma \sin(p-P) = \sin(l-L) \qquad \sin\sigma \sin\tau = \sin B$$
$$\sin\sigma \cos(p-P) = \cos(l-L)\sin B \qquad \sin\sigma \cos\tau = \cos B \sin(l-L)$$
$$\sin s = \frac{a}{\Delta} \,.\, \rho \sin\sigma \text{ or } s = a_0\nu \,.\, \rho \sin\sigma \qquad \cos\sigma = \cos B \cos(l-L).$$

The differential expressions for $\sin\sigma dp$ and $d\sigma$ become by the substitution of $dl-dN$ for du

$$\sin\sigma\, dp = \sin\tau \,.\, dl - \cos\tau \sin u \,.\, dJ + (\cos\sigma \cos p \cos\delta - \sin\sigma \sin\delta - \sin\tau) dN$$
$$d\sigma = \cos\tau \,.\, dl + \sin\tau \sin u \,.\, dJ + (\sin p \cos\delta - \cos\tau) dN.$$

But

$$\cos\sigma \cos p \cos\delta - \sin\sigma \sin\delta = \sin\tau \cos J + \cos\tau \cos u \sin J$$
$$\sin p \cos\delta = \cos\tau \cos J - \sin\tau \cos u \cos J.$$

Hence

$$\sin\sigma dp = \sin\tau \,.\, dl + [\cos\tau \cos(l-N) - \tan\tfrac{1}{2}J \sin\tau] \sin J \,.\, dN - \cos\tau \sin(l-N) dJ$$
$$d\sigma = \cos\tau \,.\, dl - [\sin\tau \cos(l-N) + \tan\tfrac{1}{2}J \cos\tau] \sin J \,.\, dN + \sin\tau \sin(l-N) dJ$$

The equations of condition between the variations of the coordinates and the variations of functions of the elements become accordingly

$$\begin{aligned}
\frac{s}{57\cdot3}\delta p = &+\nu\sin\tau.\frac{\cos\phi}{\rho}.\kappa\delta l_0 \\
&-\nu\sin\tau.\{2\cos v\}.a_0(\tan\phi+\delta e)\sin\delta(Q+N) \\
&+\nu\sin\tau.\{2\sin v\}.a_0[(\tan\phi+\delta e)\cos\delta(Q+N)-\tan\phi] \\
&+\nu\rho[\cos\tau\cos(l-N)-\tan\tfrac{1}{2}J\sin\tau].a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N \\
&-\nu\rho\cos\tau\sin(l-N).a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]
\end{aligned}$$

$$\begin{aligned}
\delta s = &+\nu\left(\cos\sigma\cos\tau.\frac{\cos\phi}{\rho}+\tan\phi\sin\sigma\sin v\right).\kappa\delta l_0 \\
&-\nu(\cos\sigma\cos\tau.\{2\cos v\}+\sin\sigma\sin v).a_0(\tan\phi+\delta e)\sin\delta(Q+N) \\
&+\nu(\cos\sigma\cos\tau.\{2\sin v\}-\sin\sigma\cos v).a_0[(\tan\phi+\delta e)\cos\delta(Q+N)-\tan\phi] \\
&-\nu\rho\cos\sigma[\sin\tau\cos(l-N)+\tan\tfrac{1}{2}J\cos\tau].a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N \\
&+\nu\rho\cos\sigma.\sin\tau\sin(l-N).a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right] \\
&+\frac{s}{a_0}.\delta a_0
\end{aligned}$$

In order to guard against incorrect deductions of the corrections of the node and inclination, which might occur, if the terms sin J . δN and δJ were kept without proper interpretation, I have substituted for

$$\kappa.\sin J.\delta N \quad .\ .\ .\ .\quad a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N$$

and for

$$\kappa.\delta J \quad .\ .\ .\ .\quad a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]$$

expressions which, as in the analogous case of the eccentricity terms, represent the true meaning of these terms, and cannot mislead. By this simple substitution all the advantages for the correctness of the deductions are secured, which otherwise might be gained by the introduction of the variations of tan $\frac{1}{2}$J sin N and tan $\frac{1}{2}$J cos N, but without their attending drawbacks, as mathematical readers may easily satisfy themselves.

The quantities

$$\kappa\delta l_0,\ a_0(\tan\phi+\delta e)\sin\delta(Q+N),\ a_0[(\tan\phi+\delta e)\cos\delta(Q+N)-\tan\phi],$$
$$a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N,\ a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]$$

which are to be treated as the unknown quantities, will in the formulæ for actual computation, of course, be represented by single letters.

For assumed circular elements the equations of condition become

$$\frac{s}{57.3}\delta p = +\nu\sin\tau\,.\,\kappa\delta l$$
$$-\nu\sin\tau\,.\cos l\,.\,2\kappa\phi\sin(Q+N)$$
$$+\nu\sin\tau\,.\sin l\,.\,2\kappa\phi\cos(Q+N)$$
$$+\nu\left[\cos\tau\cos(l-N)-\tan\tfrac{1}{2}J\sin\tau\right].\,a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N$$
$$-\nu\cos\tau\sin(l-N)\,.\,a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]$$

$$\delta s = +\nu\cos\sigma\cos\tau\,.\,\kappa\delta l_0$$
$$-\nu(\cos\sigma\cos\tau\,.\,\cos l+\tfrac{1}{2}\sin\sigma\sin l)\,.\,2\kappa\phi\sin(Q+N)$$
$$+\nu(\cos\sigma\cos\tau\,.\,\sin l-\tfrac{1}{2}\sin\sigma\cos l)\,.\,2\kappa\phi\cos(Q+N)$$
$$-\nu\cos\sigma\left[\sin\tau\cos(l-N)+\tan\tfrac{1}{2}J\cos\tau\right].\,a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N$$
$$+\nu\cos\sigma\,.\,\sin\tau\sin(l-N)\,.\,a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]$$
$$+\frac{s}{a_0}\,.\,\delta a_0.$$

In turning from the consideration of the proper treatment of measured polar coordinates to that of rectangular coordinates, it is obviously expedient to take first the special case of *Saturn's* satellites, when their rectangular coordinates parallel to the axes of the ring are either measured or estimated. In measuring the distances from the axes it is essential that the position-angles in which the measures are taken should be distinctly stated, so that if they differ from the computed position-angles of the axes which represent the fundamental plane, to which the orbits of the satellites are to be referred, the necessary corrections may be applied.

If L, B, P have the same significance as before in reference to the assumed plane of the ring, the coordinates x'', y'' parallel to the axes of the ring are

$$s\sin(p-P)=x''=a_0\nu\,.\,\rho\sin(l-L)$$
$$s\cos(p-P)=y''=a_0\nu\,.\,\rho\cos(l-L)\sin B$$

$$\delta x'' = +\nu\left[\cos(l-L)+e\cos M\right].\,\kappa\sec\phi\delta l_0$$
$$-\nu\left[\cos(l-L)(\cos\epsilon\cos\phi+\tan\tfrac{1}{2}\phi)+\cos M\right].\,a_0(\tan\phi+\delta e)\sin\delta(Q+N)$$
$$+\nu\left[\cos(l-L)\sin\epsilon\sec\phi+\sin M\right].\,a_0\left[(\tan\phi+\delta e)\cos\delta(Q+N)-\tan\phi\right]$$
$$-\nu\rho\,.\,\tan\tfrac{1}{2}J\cos(l-L)\,.\,a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N$$
$$+\frac{x''}{a_0}\,.\,\delta a_0.$$

$$\delta y'' = -\nu\sin B\left[\sin(l-L)-e\sin M\right].\,\kappa\sec\phi\delta l_0$$
$$+\nu\sin B\left[\sin(l-L)(\cos\epsilon\cos\phi+\tan\tfrac{1}{2}\phi)-\sin M\right].a_0(\tan\phi+\delta e)\sin\delta(Q+N)$$
$$-\nu\sin B\left[\sin(l-L)\sin\epsilon\sec\phi+\cos M\right].a_0\left[(\tan\phi+\delta e)\cos\delta(Q+N)-\tan\phi\right]$$
$$+\nu\rho\left[\cos B\cos(l-N)-\tan\tfrac{1}{2}J\sin B\sin(l-L)\right].\,a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N$$
$$+\nu\rho\cos B\sin(l-N)\,.\,a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]$$
$$+\frac{y''}{a_0}\,.\,\delta a_0.$$

For convenience M is put instead of $L-(Q+N)$. If the eccentric anomaly ϵ is not available

$$\frac{\cos v + e\cos\phi + \tan\frac{1}{2}\phi}{1+e\cos v} \text{ must be put for } \cos\epsilon\cos\phi + \tan\tfrac{1}{2}\phi$$

and

$$\frac{\sin v}{1+e\cos v} \text{ for } \sin\epsilon\sec\phi.$$

For assumed circular elements the equations of condition become

$$\begin{aligned}\delta x'' = &+\nu\cos(l-L)\,.\,\kappa\delta l\\ &-\nu[\cos(l-L)\cos l+\cos L]\,.\,a_0 e\sin(Q+N)\\ &+\nu[\cos(l-L)\sin l+\sin L]\,.\,a_0 e\cos(Q+N)\\ &-\nu\tan\tfrac{1}{2}J\cos(l-L)\,.\,a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N\\ &+\frac{x''}{a_0}\,.\,\delta a_0.\end{aligned}$$

$$\begin{aligned}\delta y'' = &-\nu\sin B\,.\,\sin(l-L)\,.\,\kappa\delta l\\ &+\nu\sin B[\sin(l-L)\cos l-\sin L]\,.\,a_0 e\sin(Q+N)\\ &-\nu\sin B[\sin(l-L)\sin l+\cos L]\,.\,a_0 e\cos(Q+N)\\ &-\nu[\cos B\cos(l-N)-\tan\tfrac{1}{2}J\sin B\sin(l-L)]\,.\,a_0\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\sin\delta N\\ &+\nu\cos B\sin(l-N)\,.\,a_0\left[\left(\sin J+\frac{\delta J}{\omega^\circ}\right)\cos\delta N-\sin J\right]\\ &+\frac{y''}{a_0}\,.\,\delta a_0.\end{aligned}$$

In case the x ordinates are not measured micrometrically, but estimated by comparison with certain definite points of the major axis of the ring, the proper unit for x and for the semi-axis of the satellite's orbit will obviously be the semi-diameter of *Saturn's* equator. If (a) is the length of the semi-axis expressed in such units, the computed

$$x=(a)\rho\sin(l-L)$$

and the equations of condition require merely the substitution of (a) for $a_0\,\nu$.

Though the range of such estimations of x is limited, there is the great advantage that they can be made without micrometer, and even with a rather unsteady instrument, and that when taken with due care and circumspection at the right times, they yield in accuracy only to the best micrometrical measurements. Observations which fix the times of the conjunctions of the satellites with the limbs of the ball and the ends of the ring in the four quadrants are of great value, not only in investigations

of the orbital longitudes of the satellites, but also in examining the dimensions of the ball and ring.

If the observed positions of the satellite in reference to the planet's centre consist of differences of Right Ascension and Declination, or are given in equivalent rectangular coordinates, the frequently used method for their computation may be followed, which Bessel employs in his investigation of the orbit of *Titan* ("Astr.-Nachr.," Nos. 193–195). His auxiliary angles f, F, g, G, h, H, referred to the geocentric place of the planet A, D, may be found by means of the formulæ *

$$\begin{aligned} \sin f \sin \mathrm{F} &= -\sin(\mathrm{A}-\mathrm{N}) & \sin g \sin(\mathrm{G}-\mathrm{F}) &= -\sin(\mathrm{D}+\mathrm{E}) \\ \sin f \cos \mathrm{F} &= +\cos(\mathrm{A}-\mathrm{N}) \cos \mathrm{J} & \sin g \cos(\mathrm{G}-\mathrm{F}) &= +\cos(\mathrm{D}+\mathrm{E}) \cos f \\ \cos f &= -\cos(\mathrm{A}-\mathrm{N}) \sin \mathrm{J} & \cos g &= +\cos(\mathrm{D}+\mathrm{E}) \sin f \\ \sin f \sin \mathrm{E} &= -\sin(\mathrm{A}-\mathrm{N}) \sin \mathrm{J} & \sin h \sin(\mathrm{H}-\mathrm{F}) &= +\cos(\mathrm{D}+\mathrm{E}) \\ \sin f \cos \mathrm{E} &= +\cos \mathrm{J} & \sin h \cos(\mathrm{H}-\mathrm{F}) &= -\sin(\mathrm{D}+\mathrm{E}) \cos f \\ & & \cos h &= +\sin(\mathrm{D}+\mathrm{E}) \sin f. \end{aligned}$$

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* The values of b, B, c, C in the expressions for a planet's or comet's heliocentric coordinates referred to the equator—

$$x = r \sin a \sin(\mathrm{A}' + v),$$
$$y = r \sin b \sin(\mathrm{B}' + v),$$
$$z = r \sin c \sin(\mathrm{C}' + v),$$

may be found, in a similar manner, by means of the formulæ

$$\begin{aligned} \sin a \sin \mathrm{E} &= +\sin i \cos \Omega & \sin b \sin(\mathrm{B}-\mathrm{A}) &= -\cos(\mathrm{E}+\epsilon) \\ \sin a \cos \mathrm{E} &= +\cos i & \sin b \cos(\mathrm{B}-\mathrm{A}) &= +\sin(\mathrm{E}+\epsilon) \cos a \\ \cos a &= +\sin i \sin \Omega & \cos b &= -\sin(\mathrm{E}+\epsilon) \sin a \\ \sin a \sin \mathrm{A} &= +\cos \Omega & \sin c \sin(\mathrm{C}-\mathrm{A}) &= -\sin(\mathrm{E}+\epsilon) \\ \sin a \cos \mathrm{A} &= -\sin \Omega \cos i & \sin c \cos(\mathrm{C}-\mathrm{A}) &= -\cos(\mathrm{E}+\epsilon) \cos a \\ \mathrm{A}' &= \mathrm{A} + \omega & \cos c &= +\cos(\mathrm{E}+\epsilon) \sin a \end{aligned}$$

so that the $\mathrm{B}'+v$ and $\mathrm{C}'+v$ in the expressions for y and z are obtained by simply adding B – A and C – A to the values of $\mathrm{A}'+v$.

Are these convenient formulæ for b, B, c, C not to be found anywhere in print? Considering that they refer to computations of such frequent occurrence, it is not easy to assume that they have not been pointed out before.

In case the inclination i is of moderate amount, it may be worth while, for greater accuracy, to determine A by computing

$$\sin \nu = \tan \tfrac{1}{2} i \,.\, \sin \Omega \,.\, \sin \mathrm{E}$$

which makes

$$\mathrm{A} = \Omega + 90^\circ - \nu.$$

If formulæ for control are required,

$$\sin b \sin \mathrm{B} = \sin \Omega \cos \epsilon$$
$$\sin c \sin \mathrm{C} = \sin \Omega \sin \epsilon$$
$$\tan(\mathrm{C}-\mathrm{B}) = \frac{\cos a}{\cos b \cos c}$$

may serve the purpose.

The positions of the satellite are then found by

$$\Delta_{,} \sin s \sin p = r \sin f \sin (\mathrm{F} + u) \quad = \Delta \,.\, \xi$$
$$\Delta_{,} \sin s \cos p = r \sin g \sin (\mathrm{G} + u) \quad = \Delta \,.\, y$$
$$\Delta_{,} \cos s = \Delta + r \sin h \sin (\mathrm{H} + a) = \Delta \,.\, (1 + \zeta);$$

or, if the difference between s and $\omega'' \tan s$ is neglected, the rectangular coordinates by

$$x'' = \frac{\omega''}{\Delta} \cdot \frac{r}{1+\zeta} \sin f \sin (\mathrm{F} + u) \qquad \omega'' = \frac{1}{\operatorname{arc} 1'}$$
$$y'' = \frac{\omega''}{\Delta} \cdot \frac{r}{1+\zeta} \sin g \sin (\mathrm{G} + u).$$

By the substitution of α, δ for A, D in the computations of the auxiliary angles f, F, g, G, or by computing their values, not for the apparent place of the planet, but for that of the satellite, the divisor $1+\zeta$ will be got rid of, and the formulæ become simplified. If that course is adopted they will be

$$x'' = \omega'' \sin s \sin p = \omega'' \cos \mathrm{D} \sin (\alpha - \mathrm{A}) = a_0 \nu \rho \,.\, \sin f \sin (\mathrm{F} + u)$$
$$y'' = \omega'' \sin s \cos p = \omega'' \sin (\delta - \mathrm{D}) - x'' \,.\, \tan \tfrac{1}{2}(\alpha - \mathrm{A}) \sin \delta = a_0 \nu \rho \,.\, \sin g \sin (\mathrm{G} + u)$$

or, if $\mathrm{F}' = \mathrm{F} + \mathrm{Q}$, $\mathrm{G}' = \mathrm{G} + \mathrm{Q}$,

$$x'' = a_0 \nu \sin f \,.\, \rho \sin (\mathrm{F}' + v)$$
$$y'' = a_0 \nu \sin g \,.\, \rho \sin (\mathrm{G}' + v).$$

The equations of condition are

$$\begin{aligned}
\delta x'' = & + \nu \sin f [\cos (\mathrm{F}' + v) + e \cos \mathrm{F}'] \,.\, \kappa \sec \phi \delta l_0 \\
& - \nu \sin f [\cos (\mathrm{F}' + v) (\rho \cos v \cos \phi + e \cos \phi + \tan \tfrac{1}{2} \phi) + \cos \mathrm{F}'] \,. \\
& \qquad\qquad a_0 (\tan \phi + \delta e) \sin \delta (\mathrm{Q} + \mathrm{N}) \\
& + \nu \sin f [\cos (\mathrm{F}' + v) \,.\, \rho \sin v \sec^2 \phi - \sin \mathrm{F}'] \,.\, a_0 [(\tan \phi + \delta e) \cos \delta (\mathrm{Q} + \mathrm{N}) \\
& \qquad\qquad - \tan \phi] \\
& + \nu \rho \,.\, \tan \tfrac{1}{2} \mathrm{J} \,.\, \cos (l - \mathrm{N} + \alpha - \mathrm{N}) \,.\, a_0 \left(\sin \mathrm{J} + \frac{\delta \mathrm{J}}{\omega^{\circ}} \right) \sin \delta \mathrm{N} \\
& + \nu \rho \sin (l - \mathrm{N}) \cos f \,.\, a_0 [\sin \left(\mathrm{J} + \frac{\delta \mathrm{J}}{\omega^{\circ}} \right) \cos \delta \mathrm{N} - \sin \mathrm{J}] \\
& + \frac{x''}{a_0} \delta a_0
\end{aligned}$$

$$\begin{aligned}
\delta y'' = & + \nu \sin g [\cos (\mathrm{G}' + v) + e \cos \mathrm{G}'] \,.\, \kappa \sec \phi \,.\, \delta l_0 \\
& - \nu \sin g [\cos (\mathrm{G}' + v) (\rho \cos v \cos \phi + e \cos \phi + \tan \tfrac{1}{2} \phi) + \cos \mathrm{G}'] \,. \\
& \qquad\qquad a_0 (\tan \phi - \delta e) \sin \delta (\mathrm{Q} + \mathrm{N}) \\
& + \nu \sin g [\cos (\mathrm{G}' + v) \,.\, \rho \sin v \sec^2 \phi - \sin \mathrm{G}'] \,.\, a_0 [(\tan \phi - \delta e) \cos \delta (\mathrm{Q} + \mathrm{N}) \\
& \qquad\qquad - \tan \phi] \\
& - \nu \rho [\cos (l - \mathrm{N}) \cos \delta + \tan \tfrac{1}{2} \mathrm{J} \sin (l - \mathrm{N} + \alpha - \mathrm{N}) \sin \delta] \,. \\
& \qquad\qquad a_0 \left(\sin \mathrm{J} + \frac{\delta \mathrm{J}}{\omega^{\circ}} \right) \sin \delta \mathrm{N} \\
& + \nu \rho \sin (l - \mathrm{N}) \,.\, \cos g \,.\, a_0 [\left(\sin \mathrm{J} + \frac{\delta \mathrm{J}}{\omega^{\circ}} \right) \cos \delta \mathrm{N} - \sin \mathrm{J}] \\
& + \frac{y''}{a_0} \,.\, \delta a_0.
\end{aligned}$$

If the assumed elements are circular, and if $\mathrm{F}_{,} = \mathrm{F} - \mathrm{N}$, $\mathrm{G}_{,} = \mathrm{G} - \mathrm{N}$, the coordinates become

$$x'' = a_0 \nu \sin f \,.\, \sin (\mathrm{F}_{,} + l)$$
$$y'' = a_0 \nu \sin g \,.\, \sin (\mathrm{G}_{,} + l)$$

and the equations of condition

$$\delta x'' = +\nu \sin f \cos (F_{,} + l) . \kappa \delta l_0$$
$$-\nu \sin f [\cos (F_{,} + l) \cos l + \cos F_{,}] . a_0 e \sin (Q + N)$$
$$+\nu \sin f [\cos (F_{,} + l) \sin l - \sin F_{,}] . a_0 e \cos (Q + N)$$
$$+\nu \tan \tfrac{1}{2}J . \cos (l - N + \alpha - N) . a_0 \left(\sin J + \frac{\delta J}{\omega^\circ}\right) \sin \delta N$$
$$+\nu \sin (l - N) \cos f . a_0 [\sin \left(J_0 + \frac{\delta J}{\omega^\circ}\right) \cos \delta N - \sin J]$$
$$+\frac{x''}{a_0} . \delta a_0$$

$$\delta y'' = +\nu \sin g \cos (G_{,} + l) . \kappa \delta l_0$$
$$-\nu \sin g [\cos (G_{,} + l) \cos l + \cos G_{,}] . a_0 e \sin (Q + N)$$
$$+\nu \sin g [\cos (G_{,} + l) \sin l - \sin G_{,}] . a_0 e \cos (Q + N)$$
$$-\nu [\cos (l - N) \cos \delta + \tan \tfrac{1}{2}J \sin (l - N + \alpha - N) \sin \delta] . a_0 \sin \left(J + \frac{\delta J}{\omega^\circ}\right) \sin \delta N$$
$$+\nu \rho \sin (l - N) \cos g . a_0 \sin [\left(J + \frac{\delta J}{\omega^\circ}\right) \cos \delta N - \sin J]$$
$$+\frac{y''}{a_0} . \delta a_0$$

The auxiliary angles are found by the formulæ

$$\sin f \sin E = + \sin J \sin (\alpha - N) \qquad \sin g \sin (G - F) = - \sin (E + \delta)$$
$$\sin f \cos E = + \cos J \qquad \sin g \cos (G - F) = - \cos (E + \delta) \cos f$$
$$\cos f = - \sin J \cos (\alpha - N) \qquad \cos g = + \cos (E + \delta) \sin f$$
$$\sin f \sin F = - \sin (\alpha - N)$$
$$\sin f \cos F = + \cos (\alpha - N) \cos J$$

or $F - N$ by

$$\sin (F - N + \alpha) = \tan \tfrac{1}{2}J . \cos (\alpha - N) . \sin E.$$

The values of B and P used in the formulæ for polar-coordinates are

$$\cos B \sin P = - \cos (\alpha - N) \sin J = \cos f$$
$$\cos B \cos P = + \cos (E + \delta) \sin f = \cos g$$
$$\sin B = - \sin (E + \delta) \sin f$$

On the Orbit of OΣ 400. By J. E. Gore.

Some measures of this close binary star, made in 1885 with the 23-inch Refractor of the Princeton (U.S.A.) Observatory, and kindly sent me by Professor Young, show that the companion has described about 190° of its apparent ellipse since its discovery by O. Struve in 1844.

I have computed the orbit, and find the following provisional elements:—

Elements of OΣ 400.

$P = 170{\cdot}37$ years	$\Omega = 146^\circ\ 20'$
$T = 1882{\cdot}09$	$\lambda = 43^\circ\ 30'$
$e = 0{\cdot}669$	$a = 0''{\cdot}59$
$\gamma = 36^\circ\ 58'$	$\mu = -2^\circ{\cdot}113$

The following table shows a comparison between these elements, and the observations used in the calculation of the orbit:—